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Original Article

Application of the Hamiltonian System in Deriving Solution to Dynamic System of the Sectoral Labour Market

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ABSTRACT

The paper describes the application of the Hamiltonian System in deriving a solution to the dynamic system of the sectoral labour market. The aim is to determine the best control that would allow maximum sector participation in a three-sector labour market. The three sectors of the labour market are the goods-producing sector, the service-providing sector, and the agriculture sector. The sectoral labour market is formulated as an optimal control problem whose solution is sought using the Hamiltonian system by following the necessary conditions for optimality in the Hamiltonian System of Pontryagin's Maximum Principle. The paper considered that the optimal control would be the measure of labour market efforts to produce the general worker's population, and any split would mean there is production of active sector productive workers. The sector with the highest optimal allocation of effort, defined as the control within a fixed time, is identified as the best-performing sector. Service-providing sector, in this case, presented the best control. The obtained optimal control parameter is later applied to the experimental data to check on the effect of the control on the sectoral labour market participation rate. The results of the problem indicated that the application of the control had a significant effect on the sectoral labour market participation rate as compared to when there was no control. The sector with the highest annual growth rate measured as the sectoral participation rate was identified as the goods-producing sector.

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INTRODUCTION

A Hamiltonian system (H) is described as a function of the state of a dynamic growth system at a given time. The state of the system is described by the position and the momenta (Chen et al., 2020). That is, if r is the state, x is the momenta and u is the position, then r= (x, u), and the Hamiltonian system is H (r, t). Dynamic systems are often stated as optimal control problems whose solution is sought by deriving algebraic conditions that will either be sufficient or necessary for optimality.

Pontryagin's Maximum Principle (PMP) works in the presence of constraints for the input controls applicable in the solution of an optimal control problem. The problem is changed from constrained to unconstrained by introducing a normalising value (costate /adjoint variable) greater than zero to the constrained problem. The resultant unconstrained optimal control problem is transformed by first considering an adjoint path for the optimal state and the control so as to obtain a function (the Hamiltonian) whose solution will be able to move the problem from the initial to the final state (Tedrake, 2023). The computations proceed to obtain the adjoint equation, state equation, optimality conditions and transversality condition properties that all depend on the Hamiltonian system.

Oruh and Agwu (2015) made use of the Hamiltonian Systems when deriving Pontryagin's Maximum Principle (PMP). The researchers later examined the application of PMP in comparison with Runge-Kutta methods in solving optimal control problems. The work also involved employing PMP in finding analytical solutions to formulated problems from physics, geometry, and economics. Further, a numerical approach to solving these problems was tested using the Runge-Kutta method. The results of the examination explained that the analytical solutions obtained using PMP were equivalent to numerical solutions obtained through Runge-Kutta methods. Of the two methods, error analysis showed that PMP gave a significant error compared to the Rung-Kutta methods (Oruh and Agwu, 2015).

Balseiro et al. (2017) explored simple variation splines on the Riemannian Manifold from the Hamiltonian in PMP. The aim was to minimise formulated quadratic cost functions and constant cost functions under restricted acceleration. By splitting the variables using linear connections, the solution to the optimal Hamiltonian within PMP was sought. The resulting Hamiltonian equations were studied in the case of Riemannian Manifold, and their applications were explored in computational anatomy.

Soumia et al. (2018) applied Hamiltonian formulation to be able to model and control systems widely used in applications in robotics. Using Legendre transformation of the Lagrangian equations, the researchers were able to show that Hamiltonian equations can be obtained from the second-order equations of the Langragian. The Hamiltonian formulation was validated using a manipulator with flexible joints. It was found to be able to obtain a system of Hamilton equations without symmetry, as was the case with the Langragian formalism.

Naz (2022) explored the economic growth theory of dynamic optimisation problems by employing optimal control techniques. The work aimed at the construction of the first integral for the current value Hamiltonian (CVH) systems of nonlinear ordinary differential equations for the uniform time step. The working yielded discrete time CVH for the equations in the economic growth model.

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Tarasyev et al. (2022) investigated the qualitative behaviour of solutions of Hamiltonian systems of PMP. Using the resource consumption model, the discounting factor in the model was varied to establish the phase portraits in the corresponding optimal control model at the infinite time horizon. Using the stabilisation procedure, the work established that the existence of a steady state in the vicinity of the equilibrium guarantees the stabilisation of the Hamiltonian dynamics.

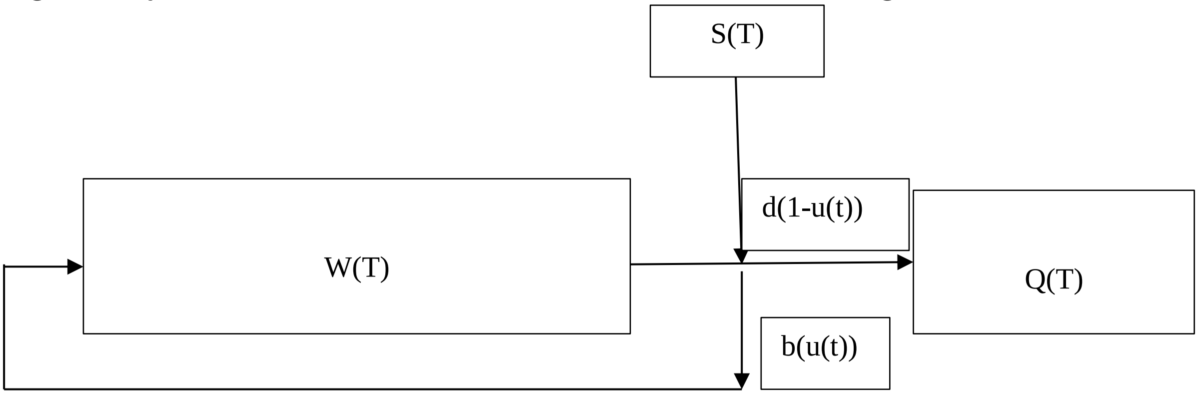
In this paper, we consider a dynamic system of Sectoral Labour market participation that changes over time. The sectoral labour market in this case, is split into goods-producing, service-providing, and agriculture sectors. By applying the Hamiltonian of Pontryagin's Maximum Principle to the Optimal Control problem of the Sectoral Labour Market Performance, we obtain the optimal controls for each sector and analyse the effects of the control on the Sectoral Labour Market participation rate in each sector.

PRELIMINARY AND PROBLEM FORMULATION

The sector model is defined by **W(t), Q(t),** and **U(t)** being the number of workers in the general labour force, the number of active workers in the sector, and the fraction of labour force efforts directed towards generating more sector active workers respectively. Maximum sector performance, in this case, is measured by sector labour force participation. That means **u(t)** is the efforts directed to generating the general labour force workers' population while **(1-u(t))** is the effort directed towards generating more sector active workers.

The model has the following assumptions: First, sector performance is measured by the number of active workers in the sector; Second, a high-performing sector has a higher fraction of labour force efforts **(1-u(t))**; Third, the performance index with **u(t)** being the control variable and **X(t)=(W(t), Q(t))** being the state variable is maximised by **J (U)** which is the maximum sectoral contribution to the overall labour market; Fourth, the model dynamics are explained by figure 1 (assuming all other factors that control the labour market remains constant); and finally, **S(t)** is the marginal contribution of the worker to the labour market performance, **b** and **d** are known transition constants. The dynamics of the sectoral labour market are derived from the general labour force, as illustrated in *Figure 1*.

Figure 1: Dynamics of the sectoral labour market as derived from the general labour force.



Assuming the population of workers W(t) and Q(t) grows exponentially, we have

$$\dot{W}(t) = -\mathrm{aw(t)} + \mathrm{bu(t)s(t)w(t)} \qquad (1)$$

$$w(0) = w^0 = 1$$

Where $\dot{W}(t)$ is the dynamics of the labour force growth per one worker, **a** is the labour force exit rate and **b** are the general labour force transition constant,

$$\dot{Q}(t) = -\mathrm{cq(t)} + \mathrm{d}(1 - \mathrm{u(t)})\mathrm{s(t)w(t)} \quad (2)$$

$$q(0) = q^0 = 1$$

Where $\dot{Q}(t)$ is the dynamics of the sectoral labour market growth per one worker, **c** is the sector

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worker exit rate, and **d** is the active sector workers transition constant.

The state variable is capital per worker **X** that is **X(T)=(W(t), Q(t))** measured as a function of the labour force efforts towards maximum sector performance. By considering optimal reinvestment of the labour force efforts to produce optimal sectoral labour market participants, the objective function yields the maximum sectoral labour market contribution, which is the sum of the production function $\boldsymbol{f}(\boldsymbol{X}(\boldsymbol{T}))$ and the integral of the sectoral participation discounted on the fixed time interval as in the equation.

$$\max_{q(t)} \; J \;\; J\,(U) = f(X(T)) + \int_0^T e^{-gt} Q(t)\, dt \qquad for\; t \;\epsilon\; [0, T] \quad (3)$$

The highest **(1-u(t))** of the sectoral labour market is sought using the Hamiltonian system. The working of the problem is guided by the following theorem. This theorem is a modified version of Theorem 1 in Aseev and Kryazhimiskiy (2005) work;

Theorem

In order that the admissible control 0≤ u(t) ≤1, 0≤ t ≤T, and the corresponding state X(T)=(W(t), Q(t)) yield a solution to the fixed interval, time-optimal control problem with fixed initial state at t=0 and unspecific final state at t=T, it is necessary that there exists a constant λ_0 and a continuous function $\lambda(t) = (\lambda_1(t), \dots, \lambda_n(t))$ such that $\lambda_0(t) \neq 0$ or $\lambda_i(t) \neq 0$ for every 0≤ t ≤T, , and that the following necessary conditions of PMP are satisfied;

i. $\dot{\lambda}_i = -\frac{\partial H}{\partial x_i}$

ii. $0 = \frac{\partial H}{\partial u}$

iii. $\dot{x}_i = \frac{\partial H}{\partial \lambda_i}$

iv. $\lambda_i(T) \geq 0$

v. $H(x(t), u, \lambda(t), t) \leq H(x^*(t), u^*(t), \lambda(t), t)$

Where conditions i, ii, iii, and iv are the adjoint equation, optimality condition, state equation and transversality conditions, respectively. The optimal control $u^*(t)$ is the value that maximises the Hamiltonian $H(x(t), u, \lambda(t), t)$ at time $t \epsilon [0, T]$. The Hamiltonian system is therefore

$$H(x, u, \lambda, \mathrm{t}) = L(x, u) + \lambda' f(x, u)$$

Proof

We shall use the variation technique to prove the four necessary conditions of the Hamiltonian System in PMP. First, we modify the objective function along the trajectory of the state equation; second, we define the optimal profile for the problem that causes a small variation of the cost function and express the condition that the variation is negative.

The solution to this problem involved finding the function U defined in [0, T] that maximises J(U), that is, when labour market sector productivity is at its maximum.

$$\max_{q(t)} \; J\,(U) = f(x(T)) + \int_0^T e^{-gt} Q(t)\, dt \qquad for\; t \;\epsilon\; [0, T]$$

Subject to constraints $\dot{x}(t) = f(x(t), u(t))$

The variation technique relies on the following analogy: If u^* is the control that maximizes J(U), then a small variation in u^* i.e. ∂u leads to a decrease in the cost function. That is

$$\partial J = J(u^* + \partial u) - J(u^*) < 0$$

If X(T)=(W(t), Q(t)) verifies the objective function along the trajectory, we modify J to $\hat{J}$ by incorporating the constraint into the function using the Langrage multiplier λ'that is

$$\hat{J} = J - \int_0^T \lambda' \, [\, \dot{x}(t) - \; f(x(t), u(t))]\mathrm{dt} \; (4)$$

The terms in the square brackets are equal to zero. Hence $\hat{J} = J$ and if u^* Optimises J it will also optimise $\hat{J}$.

Now, define the Hamiltonian and rewrite the objective function using the definition of the Hamiltonian

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if

$H(x,u,\Lambda,\mathrm{t}) = L(x,u) + \lambda' f(x,u)$

And

$\lambda' f(x,u) = L(x,u) - H(x,u,\Lambda,\mathrm{t})$

Then, the objective function using equation 4 becomes

$\hat{J} = J - \int_0^T \lambda' \,[\,\dot{x}(t) - f(x(t),u(t))]\,\mathrm{dt}$

Which is

$\hat{J} = f(x(T)) + \int_0^T e^{-gt} Q(t)\,dt - \int_0^T \lambda' \,\dot{x}(t)dt + \int_0^T H(x,u,\Lambda,\mathrm{t})\,dt - \int_0^T e^{-gt} Q(t)\,dt$

And hence

$\hat{J} = f\big(x(T)\big) + \int_0^T [H(x,u,\Lambda,\mathrm{t}) - \lambda' \,\dot{x}(t)]dt$ (5)

Next, we define the optimal profile of the problem.

Let $\{u(t), 0 \le t \le T\}$ be the optimal control function that together with the initial conditions, determines the state function along the optimal profile $\{x(t), 0 \le t \le T\}$. A small variation along the optimal profile $(x(t) + \partial x(t))$ will lead to a small change in the optimal control $(u(t) + \partial u(t))$ lets call this v. The change is so small in that for all u_i and v_i,

$\int_0^T |u_i(t) - v_i(t)|dt < \varepsilon$ where ε is a real number, meaning that v is closer to u

Let ∂J be the corresponding variation in the objective function that is

$\partial\hat{J} = \hat{J}(v) - \hat{J}(u) < 0$

If u is optimal, then this variation must be negative since u, by definition, maximises $\hat{J}$.

The modified objective function in equation (5), written using the variation, becomes

$\partial\hat{J} = f\big(x(T) + \partial x(T)\big) + \int_0^T [H(x + \partial x, v, \Lambda,\mathrm{t}) - \lambda'(\,\dot{x}(t) + \partial\dot{x}(t))]\,dt - f\big(x(T)\big) - \int_0^T [H(x,u,\Lambda,\mathrm{t}) - \lambda' \,\dot{x}(t)]\,dt$

This gives

$\partial\hat{J} = f\big(x(T) + \partial x(T)\big) - f\big(x(T)\big) + \int_0^T [H(x + \partial x, v, \Lambda,\mathrm{t}) - H(x,u,\Lambda,\mathrm{t}) - \lambda' \partial\dot{x}(t)]\,dt$

Using integration by parts, we eliminate the variation of the derivative of the state $\lambda' \partial\dot{x}(t)$

$\int_0^T \lambda' \partial\dot{x}(t)\,dt = \lambda'(T)\partial x(T) - \lambda'(0)\partial x(0) - \int_0^T \dot{\lambda}' \partial x(t)dt$

Since the control variation does not change the problem's initial conditions, the $\partial x(0) = 0$

And the derivation becomes

$\int_0^T \lambda' \partial\dot{x}(t)\,dt = \lambda'(T)\partial x(T) - \int_0^T \dot{\lambda}' \partial x(t)dt$

And

$\partial\hat{J} = f\big(x(T) + \partial x(T)\big) - f\big(x(T)\big) - \lambda'(T)\partial x(T) + \int_0^T [H(x + \partial x, v, \Lambda,\mathrm{t}) - H(x,u,\Lambda,\mathrm{t}) + \dot{\lambda}' \partial x(t)]\,dt$

Using the Taylor series, the 1st order approximation of the terminal state and the Hamiltonian with respect to the state becomes

$f\big(x(T) + \partial x(T)\big) \approx f\big(X(T)\big) + f_x(x(T))\partial x(T))$

$H(x + \partial x, v, \Lambda,\mathrm{t}) \approx H(x,v,\Lambda,\mathrm{t}) + H_x(x,v,\Lambda,\mathrm{t})\partial x$

This helps us to conclude that

$\partial\hat{J} = f\big(x(T)\big) + f_x(x(T))\partial x(T)) - f\big(x(T)\big) - \lambda'(T)\partial x(T) + \int_0^T [H(x,v,\Lambda,\mathrm{t}) + H_x(x,v,\Lambda,\mathrm{t})\partial x - H(x,u,\Lambda,\mathrm{t}) + \dot{\lambda}' \partial x(t)]\,dx$

Therefore, first-order terms become

$\partial\hat{J} = [f_x(x(T) - \lambda'(T)]\partial x(T) + \int_0^T [H_x(x,v,\Lambda,\mathrm{t}) + \dot{\lambda'}]\partial x(t) + H(x,v,\Lambda,\mathrm{t}) - H(x,u,\Lambda,\mathrm{t})]\,dx$

Next, we select λ that satisfies the adjoint equation and the transversality condition. i.e

$-\dot{\lambda}'(t) = H_x(x,v,\Lambda,\mathrm{t})$

With the final condition being

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$\lambda'(T) = f_x(x(T))$

The expression for the variation of the objective function equation becomes

$$\partial\hat{J} = \int_0^T H(x, v, \Lambda, \mathrm{t}) - H(x, u, \Lambda, \mathrm{t})]\, dx$$

As defined earlier, we know that. $\partial\hat{J}$ is negative. The conclusion therefore is that if u is optimal at any instant t, then

$$H(x, v, \Lambda, \mathrm{t}) \leq H(x, u, \Lambda, \mathrm{t})$$

THE HAMILTONIAN APPLICATION TO LABOUR MARKET SECTOR PERFORMANCE MODEL

The Hamiltonian considers a system with known end time **T**, known starting point but with free endpoints. Thus, the Hamiltonian is defined by the time. The Hamiltonian applies in obtaining an optimal control $\boldsymbol{u}^*(.)$ that will take the system to the desired production (state), that is, when maximum production of the system will be achieved.

Solving a time-optimal control problem does not require the use of a cost function, hence the solution for this problem will not have the Langragian.

Now, considering equations (1) and (2) the Hamiltonian is expressed as;

$$H = \lambda_1 \dot{w}(t) + \lambda_2 \dot{Q}(t) \qquad (6)$$

Which in this case is

$$H = \lambda_1[-\mathrm{aw(t)} + \mathrm{bs(t)u(t)w(t)}] + \lambda_2[-\mathrm{cq(t)} + \mathrm{d}(1 - \mathrm{u(t)})\mathrm{s(t)w(t)}]$$

The optimality conditions for the model were;

i. $\frac{\partial H}{\partial w} = -\dot{\lambda}_1 = -\lambda_1(\mathrm{bs(t)u(t)} - a) - \lambda_2 \mathrm{d}(1 - \mathrm{u(t)})\mathrm{s(t)}$

ii. $\frac{\partial H}{\partial q} = -\dot{\lambda}_2 = \lambda_2 \mathrm{c}$

iii. $\frac{\partial H}{\partial u} = \lambda_1(\mathrm{bs(t)w(t)}) + \lambda_2(-\mathrm{ds(t)w(t)} = 0$

iv. $\frac{\partial H}{\partial \lambda_1} = \dot{w}(t) = -\mathrm{aw(t)} + \mathrm{bs(t)u(t)w(t)}$

v. $\frac{\partial H}{\partial \lambda_2} = \dot{q}(t) = -\mathrm{cq(t)} + \mathrm{d}(1 - \mathrm{u(t)})\mathrm{s(t)w(t)}$

Setting $\frac{\partial H}{\partial w} = \frac{\partial H}{\partial q} = \frac{\partial H}{\partial u} = 0$

Optimality condition (i) becomes;

$$\frac{\partial H}{\partial w} = \dot{\lambda}_1 = -\lambda_1(\mathrm{bs(t)u(t)} - a) - \lambda_2 \mathrm{d}(1 - \mathrm{u(t)})\mathrm{s(t)} = 0$$

Solving for $\mathrm{u(t)}$ to obtain the expression for the optimal control

$$-\lambda_1 \mathrm{bs(t)u(t)} + \lambda_1 a - \lambda_2 \mathrm{ds(t)} + \lambda_2 \mathrm{du(t)s(t)} = 0$$

$$\mathrm{u(t)}(\lambda_2 \mathrm{ds(t)} - \lambda_1 \mathrm{bs(t)}) = \lambda_2 \mathrm{ds(t)} - \lambda_1 a$$

$$u^*(t) = \frac{\lambda_2 \mathrm{ds(t)} - \lambda_1 a}{\lambda_2 \mathrm{ds(t)} - \lambda_1 \mathrm{bs(t)}}$$

Optimality condition (iii) becomes;

$$\frac{\partial H}{\partial u} = \lambda_1(\mathrm{bs(t)u(t)w(t)}) - \lambda_2 \mathrm{ds(t)w(t)} = 0$$

$$\lambda_1(\mathrm{bs(t)w(t)} = \lambda_2 \mathrm{ds(t)w(t)}$$

$$\lambda_2 = \frac{b\lambda_1}{d}$$

We need to determine the values λ_1 and λ_2 using $\dot{\lambda}_1$ and $\dot{\lambda}_2$. To obtain λ_1 using condition (i)

$$\dot{\lambda}_1 = -\lambda_1(\mathrm{bs(t)u(t)} - a) - \lambda_2 \mathrm{d}(1 - \mathrm{u(t)})\mathrm{s(t)}$$

But $\lambda_2 = \frac{b\lambda_1}{d}$

By substituting λ_2 into $\dot{\lambda}_1$ we obtain

$$\dot{\lambda}_1 = \lambda_1(a - \mathrm{bs(t)})$$

For this part we focus on finding $\boldsymbol{\lambda_1}$ and $\boldsymbol{\lambda_2}$ that maximises **(1-u(t))**, that is, where the efforts directed towards sector participant's generation is at its maximum that is where **(1-u(t))** is greater than **u**.

By use of differential equations analogy where the derivative of a function is proportionate with the size of function. The proportionality function in this case is $(\boldsymbol{a} - \mathbf{bs}(\mathbf{t}))$. Therefore,

$$\lambda_1 = AExp\, t(a - \mathrm{bs(t)})$$

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Where **A** is the population size at time **t=0**, where in the proposed model $w(0) = w^0 = 1$

This indicates that

$\lambda_1 = Exp\ t(a - \mathrm{bs(t)})$

Optimality condition (ii) to obtain $\boldsymbol{\lambda_2}$;

$\dot{\lambda}_2 = -\lambda_2 \mathrm{c}$

Using the same analogy with the proportionate constant being $\boldsymbol{c}$,

$\lambda_2 = \mathrm{BExp\ t(-c)}$

Where $\boldsymbol{B}$ is the population size at time $\boldsymbol{t = 0}$, where in the proposed model $q(0) = q^0 = 1$

This indicates that

$\lambda_2 = \mathrm{Exp\ t(-c)}$

Optimality condition (iv) and (v) were used in obtaining the unknown parameters which are $\{a, b, \mathrm{c\ and\ d}\}$. The constants were attained by fitting the experimental data into the pairs of simultaneous equations $\frac{\partial H}{\partial \lambda_1}$ and $\frac{\partial H}{\partial \lambda_2}$.

The solution to the optimal control problem is computed as a function of the parameters hence determining the sector with the best optimal control.

$$u^*(t) = \begin{cases} 1 & t = 0 \\ \frac{\lambda_2 \mathrm{ds(t)} - \lambda_1 a}{\lambda_2 \mathrm{ds(t)} - \lambda_1 \mathrm{bs(t)}} & t \epsilon [0, T] \end{cases}$$

The model with the optimal control therefore becomes

$$\dot{w}(t) = -\mathrm{aw(t)} + \mathrm{b}u^*(t)\mathrm{s(t)w(t)} \qquad (7)$$

$$\dot{q}(t) = -\mathrm{cq(t)} + \mathrm{d}\big(1 - u^*(t)\big)\mathrm{s(t)w(t)} \quad (8)$$

HAMILTONIAN EFFECT ON SECTOR LABOUR MARKET PARTICIPATION

This section considers the application of the Hamiltonian to the situation whereby there is a split of efforts between producing more general workers' population and producing active sector participants. That is the general labour force efforts to reinvest into producing more workers or generate the active sector, productive workers. The sector with the highest optimal allocation of efforts to produce more sector participants within a fixed time therefore is considered to be the best performing sector.

The data used to validate the model was obtained from labour Organization Statistics data of 2019. The data is organised in labour market indicators that include information on annual labour force population, employment by economic activity divided into 3 sectors: output per worker, unemployment, employment by occupation, and external factors to the labour market. The period of analysis is between year 2010 and 2019.

In order to compare the different sectors optimal efforts allocation, the obtained optimal control equations (7) and (8) were used for quantitative prediction of the sector participation by considering that it depended on time. The season length is measured as a unit by considering time from year one to year ten. The normalising factor remains that $\boldsymbol{0 \leq u(t) \leq 1}$. Any $\mathbf{u(t)}$ that is below zero and above one is considered an outlier.

To be able to view these results numerically, the obtained controls together with the model coefficients **a, b, c** and **d** were applied into equations (1), (2), (7), and (8), to obtain the sectoral labour force participating population. This in turn is converted into a rate with the equation being:

$$growth\ rate\ of\ x_n = \left(\frac{x_n - x_{n-1}}{x_n}\right) \times 100 \quad (9)$$

Where; x_n - the current year sector participating population; x_{n-1} -the previous year sector participating population; n -the number of years between year 1 and year 10.

The list of parameters obtained from fitting experimental data on the Hamiltonian conditions, the adjoint values, the optimal control together with the model coefficients are tabulated in *Table 1*.

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Table 1: Table presenting the obtained model parameters after fitting the experimental data into the Hamiltonian conditions of the sectoral labour market model

Parameter	Description	Parameter Value		
		Goods Producing Sector	Service Providing Sector	Agriculture Sector
Lamda 1	Adjoint response to the general labour force growth	0.8467	0.7747	0.7328
Lamda 2	Adjoint response to sector growth	0.7566	0.1336	0.6114
$u^*(t)$	Optimal efforts allocated to developing the general workforce population	0.274	0.083	0.1797
1-$u^*(t)$	Optimal efforts allocated towards sector development	0.726	0.916	0.8203
a	Labour force exit rate	0.0329	0.00832	0.00401
b	General labour force transition constant	0.001815	0.0174	0.0332
c	Sector worker exit rate	0.05578	0.2516	0.0492
d	Sector workers transition constant	0.02279	0.048	0.0159

Table 1 shows that the obtained model controls given by (1-$u^*(t)$) indicates that service providing sector had the best sectoral labour productivity, followed by the Agricultural sector then goods producing sector at a control of 0.916, 0.8203, and 0.726 respectively.

Numerical Results of the Model

The dynamics of the model in equation (1) and (2) indicate that $\frac{\partial H}{\partial \lambda_1} = \dot{w}(t)$ and $\frac{\partial H}{\partial \lambda_2}=\dot{Q}(t)$ without the optimal control presents the participation growth rates as shown in *Figure 2*;

Figure 2 indicates that the rate of sector participants' population increase varies from one sector to the other as shown in charts (a) to (c). Chart (d) which shows the comparison in their growth rate without the control indicates that goods producing and agriculture sectors presents steady growth while service providing sectors have steady growths but with negative and positive spikes in years 6 and 7 respectively.

Application of the obtained optimal control values $u^*(t)$ and $(1 - u^*(t))$ to the model dynamics in equations (7) and (8) indicate that there are significant changes in the sector participation associated with the involvement of the Hamiltonian system as shown in *Figure 3*.

Figure 3, chart (a) indicates that the number of goods producing sector participants with the optimal control is higher as compared to that of the model without the control. Chart (b) shows that the service providing sector displays similar results to those of goods producing sectors in that the problem with the control gives higher participation compared to when there is no control. The optimal control model for the two sectors indicates that; although the growth rate could be fluctuating over time for the goods producing sector, the lowest in the controlled environment does not get to when there is no control. The service providing sector growth does not have notable fluctuations in the optimal control model, it seems to be normalised at a higher level as compared to the model with no control. Chart (c) indicates that Agriculture sector participants' growth rate on the other hand is normalised within that of the model with no optimal control. Chart (d) presents the comparison chart where goods producing sector has the highest participation rate followed by service providing then agriculture sector.

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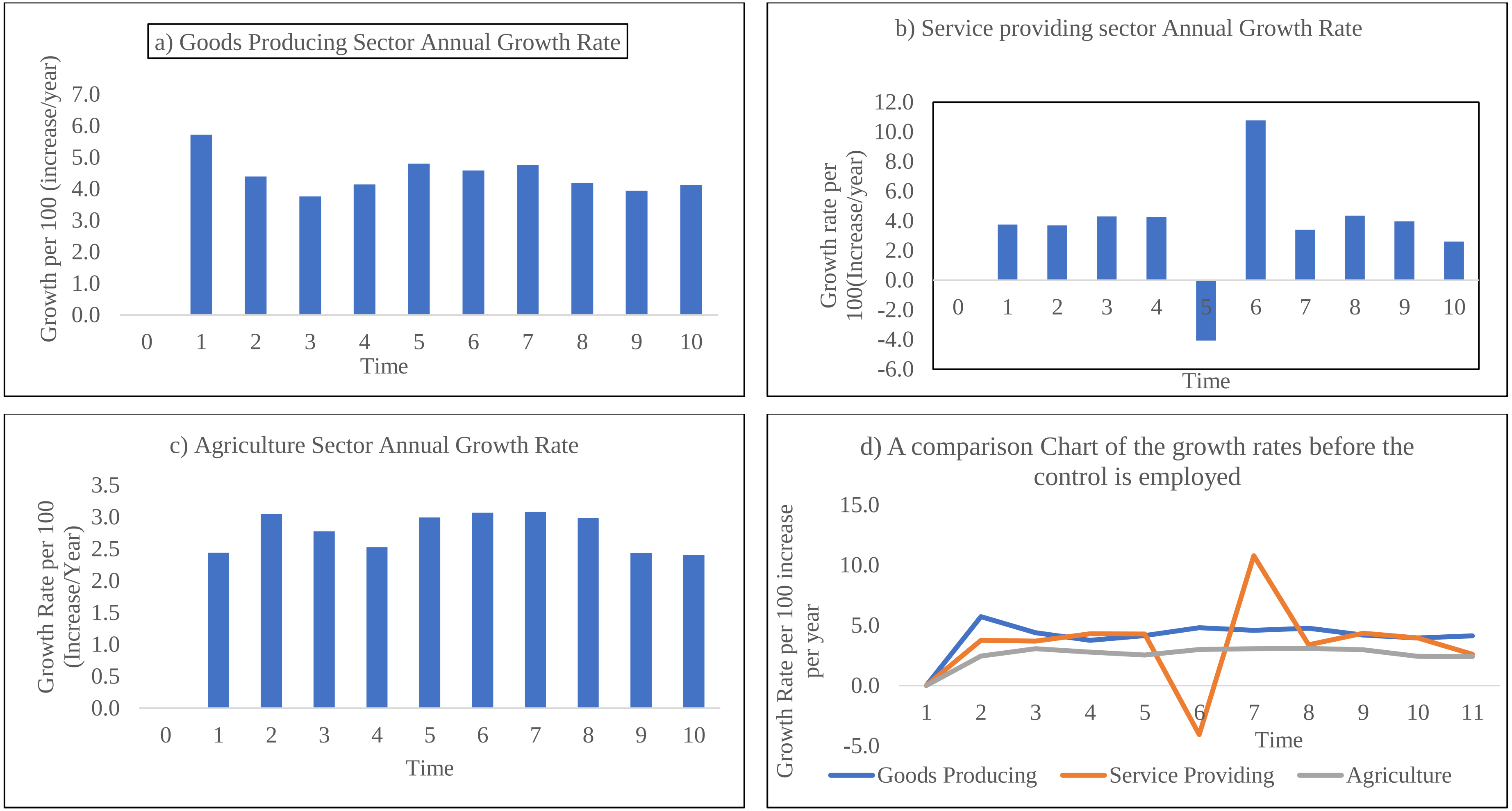
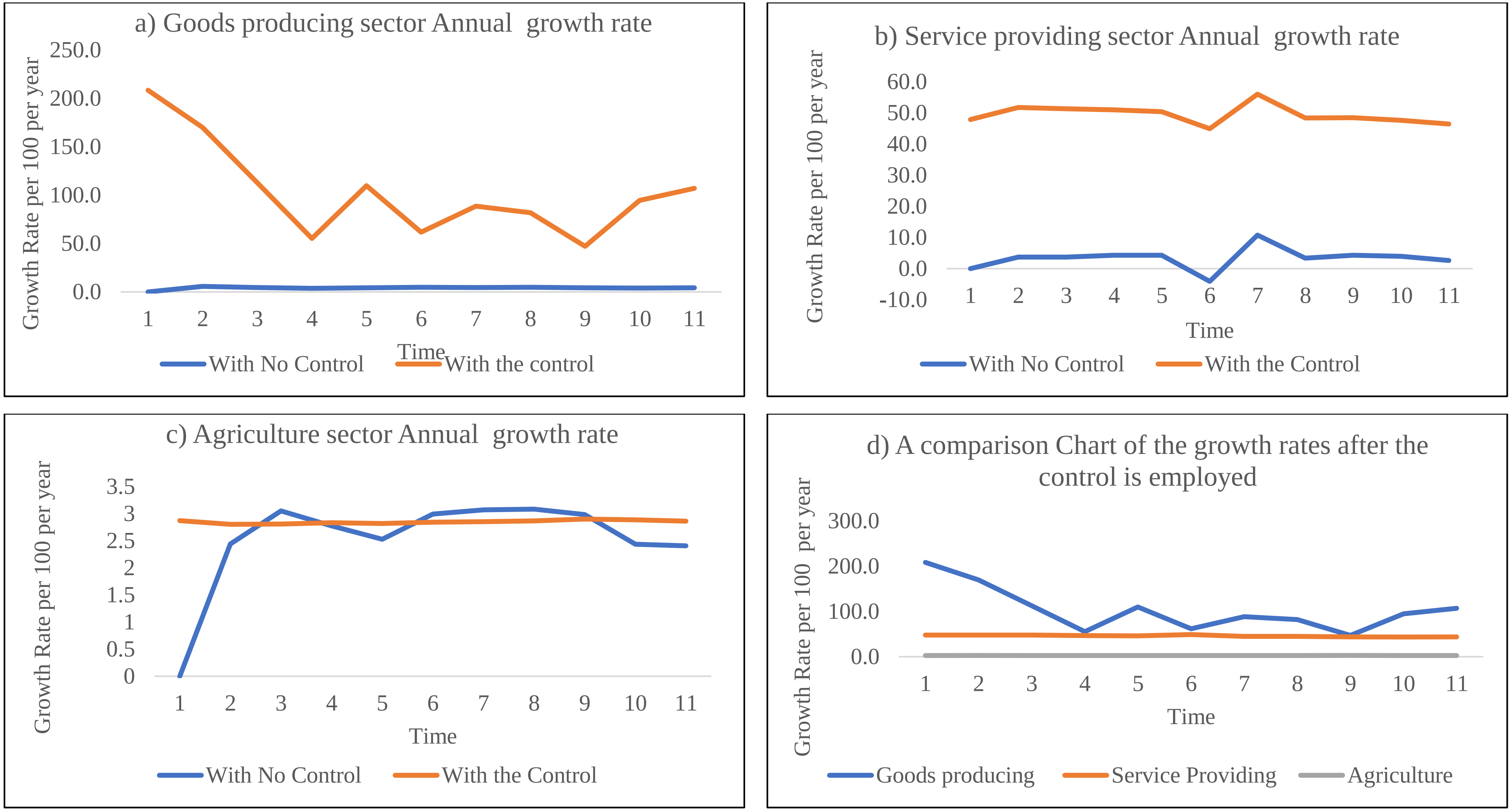
Figure 2: A collection of charts showing the sectoral growth rates and their comparisons before the optimal control is applied. The experimental years are year 1 to 10 which are between 2010 to 2019.



Article DOI: https://doi.org/10.37284/ijar.6.1.1391

Figure 3: Figure 3. A collection of charts of rate of sector participation growth in goods producing, service providing, and Agriculture Sectors together with their comparison chart. The experimental years are year 1 to 10 which are between 2010 to 2019



Article DOI: https://doi.org/10.37284/ijar.6.1.1391

CONCLUSION

This paper discusses the necessary optimality conditions when solving a dynamic system of the sectoral labour market problem using the Hamiltonian of Pontryagin's maximum principle. The conditions assist in seeking solutions to the model parameters as well as the best control of the specific sector. The results of the problem indicate that the application of the control have a significant effect in the sectoral labour market participation as compared to when there is no control. Of the three sectors, service providing sector has the highest labour participation rate though goods producing sector has the highest growth rate compared to the rest of the sectors. This paper recommends the application of the Hamiltonian system of the Pontryagin's Maximum Principle in obtaining solution to dynamic problems of the labour market. Further research can be explored in applying Extended Hamiltonian, Maximum Extended Hamiltonian, and Maximum Hamiltonian to Labour Market problems.

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